Written Round: February 25-27, 2022 at your school

Ciphering Round: April 9, 2022 at University of North Alabama

COMPREHENSIVE EXAMINATION

Construction of this test directed
by
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INSTRUCTIONS

This test consists of 50 multiple choice questions. The questions have not been arranged in order of di culty. For each question, choose the best of the ve answer choices labeled A, B, C, D and E.

The test will be scored as follows: 5 points for each correct answer, 1 point for each question left unanswered and 0 points for each wrong answer. (Thus a \perfect paper" with all questions answered correctly earns a score of 250, a blank paper earns a score of 50, and a paper with all questions answered incorrectly earns a score of 0.)

Random guessing will not, on average, either increase or decrease your score. However, if you can eliminate

Why Major in Mathematics?

What sorts of jobs can I get with a mathematics degree? Examples of occupational opportunities available to math majors:

- Market Research Analyst
- Air Tra c Controller
- Climate Analyst
- Estimator
-

1. The quartic polynomial $3x^4 \quad 8x^3 \quad 50x^2 \quad 57x \quad 18$ can be factored as $(x^2 \quad 5x \quad 6)(ax^2 + bx + c)$. Find $a + b + c$.

   (A) 23   (B) 7   (C) $\boxed{13}$ (D) 27   (E) None of these

2. Two roots of the polynomial $3x^3 + ax^2 \quad 5x + 10$ are r and r for r a real number. What is the value of a?

   (A) $\boxed{6}$ (B) 6   (C) 4   (D) 4   (E) None of these

3. Two sides of a triangle measure 8 inches and 12 inches. If the altitude to the 8 inch side is 6 inches, what is the altitude to the 12 inch side?

   (A) $\boxed{4}$ (B) 6   (C) 8   (D) 10   (E) None of these

4. Find the value of $1^2 \quad 2^2 + 3^2 \quad 4^2 + 5^2 \quad 6^2 + \quad + 19^2 \quad 20^2$.

   (A) 13175   (B) $\boxed{210}$ (C) 1330   (D) 2870   (E) None of these

5. An isosceles triangle has two sides of length 10 and one of length 8. What is the area of the triangle?

   (A) 40   (B) 80   (C) $4\sqrt{21}$ (D) $\boxed{8\sqrt{21}}$ (E) None of these

6. A history nal exam has 25 questions. If the questions were all equally weighted, Norman would have earned a 72%. However, the rst twenty questions are worth three points each, and the nal ve questions are worth eight points each. If Norman got a 64% on the test with this point distribution, how many eight point questions did Norman get correct?

   (A) 1   (B) $\boxed{2}$ (C) 3   (D) 4   (E) None of these

7. The graph of the cubic function $f(x) = ax^3 + bx^2 +$

11. Find the sum of all solutions to the equation $\cos(3x) + \cos(2x) + \cos(x) = 0$ on the interval $[0; 2\pi)$.

(**A**) 2π (**B**) 3π (**C**) 5π (**D**) $\boxed{6\pi}$ (**E**) None of these

12. Let a and b be real numbers, with $a \neq 0$. The quadratic equation $a^2 x^2$

21. Find the product of the largest and smallest solutions of the equation $(2x^2 + 4x - 3)^2 = 9 - 4x - 2x^2$.

(A) -6 (B) -3 (C) $\boxed{-\dfrac{5}{2}}$ (D) 0 (E) None of these

22. In triangle ABC, the interior angle at vertex B measures $72°$, and the exterior angle at vertex A measures $145°$. What is the measure of the interior angle at vertex C, in degrees?

(A) 37 (B) 45 (C) 54 (D) $\boxed{73}$ (E) None of these

23. For what values of $x > 1$ does the inequality $x^2 \leq (\ln x)^{\ln x}$ hold?

(A) $x \geq \dfrac{e}{e}$ (B) $x \geq e$ (C) $x \geq e^{2e}$ (D) $\boxed{x \geq e^{(e^2)}}$ (E) None of these

24. A jar contains 10 Red, 8 Purple, and 7 Blue marbles. If you draw two marbles without replacement, what is the probability you get one red and one blue?

(A) $\dfrac{14}{125}$ (B) $\dfrac{83}{120}$ (C) $\dfrac{7}{60}$ (D) $\boxed{\dfrac{7}{30}}$ (E) None of these
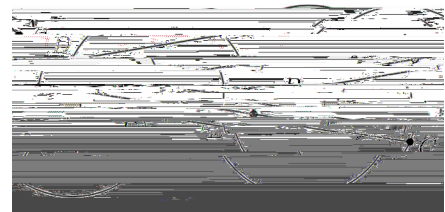
25. For $f(x)$ defined as the piecewise function below, find the sum of all real values of x for which $f(x) = 0$.

$$f(x) = \begin{cases} |x + 3| & \text{for } x \leq -1 \\ x^2 + 3x - 4 & \text{for } -1 < x < 2 \\ \log_5(\frac{1}{5}x) & \text{for } x \geq 2 \end{cases}$$

(A) -6 (B) -2 (C) -1 (D) $\boxed{3}$ (E) None of these

26. In the figure shown, the measure of angle $\angle DBE$ is $38°$, and the measure of the minor arc DE is $40°$. What is the measure of minor arc AC?

(A) 36 (B) 39 (C) 78

(D) $\boxed{116}$ (E) None of these

27. Find the smallest solution to the equation $\log_2(x) + \log_{\frac{1}{2}}\dfrac{1}{x+1} = \log_8(27)$.

(A) $\boxed{\dfrac{1 + \sqrt{13}}{2}}$ (B) $\dfrac{1 - \sqrt{13}}{2}$ (C) $\dfrac{1 + \sqrt{37}}{2}$ (D) $\dfrac{1 - \sqrt{37}}{2}$ (E) None of these

28. What is the largest value of y for which a pair (x, y) satisfies $4x^2 + 8y^2 = 4x + 16y + 9$?

(A) $\dfrac{1}{2}$ (B) $\dfrac{5}{4}$ (C) $\boxed{\dfrac{5}{2}}$ (D) $\dfrac{13}{4}$ (E) None of these

29. If $a + b = 4$, and $a^2 + b^2 = 12$, then what is $a^4 + b^4$?

(A) 112 (B) $\boxed{136}$ (C) 144 (D) 256 (E) None of these

30. Find the radius in units of a sphere for which its volume in cubic units is equal to twice its surface area in square units.

(A) 2   (B) 3   (C) 4   (D) $\boxed{6}$ (E) None of these

31. If $f(x) = \dfrac{x^{26} + x^{24} + 2x^{22}}{x^2 - 1}$, find $f(i)$, where i is the imaginary unit.

(A) $1 - i$ (B) $-1 + i$ (C) $-1 - i$ (D) $\boxed{1 + i}$ (E) None of these

32. In triangle ABC given, points D and E lie on sides $\overline{AB}$ and $\overline{BC}$, respectively, with $AC = 3$, $CE = 3$ and $BE = 6$. Given that $\angle ACB$ and $\angle DEB$ are both right angles, find $\cos(\angle ADE)$.

(A) $\dfrac{3\sqrt{10}}{10}$ (B) $\boxed{-\dfrac{\sqrt{10}}{10}}$ (C) $\dfrac{\sqrt{10}}{10}$

(D) $-\dfrac{3\sqrt{10}}{10}$ (E) None of these

33. If $f(x) = \log\dfrac{1 + x}{1 - x}$, for $-1 < x < 1$, write $f\left(\dfrac{3x + x^3}{1 + 3x^2}\right)$ in terms of $f(x)$.

(A) $f(x^3)$ (B) $[f(x)]^3$ (C) $f(x + 3)$ (D) $f(3x)$ (E) $\boxed{3f(x)}$

34. A harshad number is an integer number which is divisible by the sum of its digits. Which of the following is NOT a harshad number?

(A) 2022   (B) 2023   (C) 2024   (D) 2025   (E) $\boxed{2026}$

40. If $\sin + \cos = \frac{1}{2}$, what is the value of $\sin^2 \cos^2$?

(A) 1   (B) $\dfrac{1}{144}$ (C) $\dfrac{1}{64}$ (D) $\boxed{\dfrac{9}{64}}$ (E) None of these

41. A line through the points $(m, 9)$ and $(7, m)$ has slope m. Find the value of m.

(A) 3   (B) 2   (C) 2   (D) $\boxed{3}$ (E) None of these

42. The roots of the equation $x \quad 10 = \dfrac{24}{x}$ represent two sides of a scalene triangle. Which of the following would be a valid third side to make the triangle an acute triangle?

(A) 1   (B) 3   (C) 9   (D) 11   (E) $\boxed{\text{None of these}}$

43. In the gure shown, point O is the center of the circle and A, B and C

49. Two identical circles are placed into a square in such a way that they are tangent to each other at a single point, and each circle is tangent to the square at two points, as shown. If the radius of each circle is 1, what is the area of the square?

(A) $\dfrac{25}{2}$ (B) $\dfrac{49}{4}$ (C) $3 + 2\sqrt{2}$ (D) $\boxed{6 + 4\sqrt{2}}$ (E) None of these

50. The equation $2x + 3y - 4z = 5$ has infinitely many integer solutions. Which of the following cannot be a value of y for any integer values of x and z?

(A) $\boxed{2}$ (B) 3   (C) 5   (D) 9   (E) None of these